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ON THE FAILURE OF STATIC STABILITY ANALYSES OF NONCONSERVATIVE SYSTEMS IN REGIONS OF DIVERGENCE INSTABILITY

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Abstract—The stability of perfect bifurcational discrete dissipative systems under follower loads in regions of existence/non-existence of adjacent equilibria is thoroughly re-examined in the light of recent progress in nonlinear dynamics. A general theory for such nongradient systems described by autonomous ordinary differential equations is developed. Conditions for the existence of adjacent equilibria, the stability of precritical, critical and postcritical states, as well as for different types of local bifurcations are established. Focusing attention on the interaction of geometric nonlinearities and vanishing damping, new findings contradicting widely accepted results of the classical (linear) analysis are discovered. In a small region of adjacent equilibria near a compound branching point, which is explicitly determined, an interaction of two consecutive postbuckling modes occurs related to the following phenomena: in case of vanishing damping, loss of stability may occur via a Hopf (dynamic) bifurcation prior to static (divergence) buckling. Moreover, the critical states of divergence instability may be associated with a double zero Jacobian eigenvalue satisfying also the conditions of a Hopf (local) bifurcation. Besides local (dynamic) bifurcations, global bifurcations are also found. An example is used to illustrate the qualitative findings.

1. INTRODUCTION

There has been a large amount of work in the last 30 years on nonconservative systems under follower loading losing their stability either by divergence (static instability) or by flutter (dynamic instability). While the conditions for instability in the first case can be obtained by using either the static or the kinetic criterion, in the second case they are established only by employing the kinetic criterion (Bolotin, 1963; Herrmann and Bungay, 1964; Ziegler, 1968; Leipholtz, 1970; Plaut, 1976; Kounadis, 1977; Huseyin, 1978; Kounadis, 1983). The intent of the paper is not to survey pertinent studies but to re-examine the validity of various widely recognized findings based mainly on classical analyses in the light of recent progress in nonlinear dynamics. To this end particular attention is paid to the coupling effect of geometric nonlinearities and damping since the precise modelling of any real structural system must include both these parameters.

This investigation deals with autonomous damped or undamped multi-parameter discrete systems under partial follower loading which may lose their stability either by flutter or by divergence associated with a branching point emanating from a trivial precritical path. In a very recent paper referring to nonconservative systems with precritical deformations it was shown (Kounadis, 1992a) that the static (limit point) critical load is always higher than the dynamic buckling load regardless of the amount of damping and mass distribution. Hence, the actual load-carrying capacity of the latter systems can be established only by a nonlinear dynamic analysis.

The main objectives of this work, referring to perfect bifurcational systems without precritical deformation, are to clarify the following questions:

(a) Are there Hopf or other types of local or global dynamic bifurcations in regions of existence of adjacent equilibria?
(b) Is it possible for a system to lose its stability via flutter (dynamic instability) in a region of adjacent equilibria?
(c) Are there regions of adjacent equilibria where the static criterion fails to predict the actual (divergence) buckling load?

(d) Does the mass distribution and amount of damping influence the critical load and stability in regions of adjacent equilibria?
(e) When do the linearized equations of motion lead to results identical with those of the original equations of motion?

A general qualitative theory is comprehensively developed leading to a successful discussion of the above questions. In the framework of present considerations several new findings are obtained as by-products which invalidate widely accepted results based on classical stability analyses. The theory and its findings are illustrated using Ziegler's two-degree-of-freedom model for which many numerical results are available.

The paper deals essentially with structural systems, and as such is not addressed to scientists familiar with the rudiments of the mathematical theory of dynamical systems, but mainly to structural engineers. Because of this, some care has been taken to include in the text explanatory statements concerning certain basic concepts from the above theory for the purpose of making the analysis more comprehensible.

2. MATHEMATICAL FORMULATION

Nongradient, geometrically perfect, discrete structural damped systems with trivial precritical equilibrium paths are considered. For such a system under a partial follower compressive force λ (of constant magnitude) associated with a nonconservativeness parameter η, Lagrange equations of motion in terms of generalized displacements q_i and generalized velocities $\dot{q}_i$ $(i = 1, \ldots, n)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_i}\right)-\frac{\partial K}{\partial q_i}+\frac{\partial U}{\partial q_i}+\frac{\partial F}{\partial \dot{q}_i}-Q_i=0 \quad (i=1,\ldots,n) \tag{1}$$

where the dots denote differentiation with respect to time t; $K = (1/2)a_{ij}\dot{q}_i\dot{q}_j$ is the positive definite function of the total kinetic energy; $U = U(q_i)$ is the positive definite function of the strain energy, being a nonlinear analytic function of q_i; $F = (1/2)c_{ij}\dot{q}_i\dot{q}_j$ is the non-negative definite (viscous) dissipative function of Rayleigh; $Q_i = Q_i(q_i; \eta; \lambda)$ designate generalized, in general, nonpotential forces being nonlinear analytic functions of q_i and η, and linear functions of λ. Obviously, the tensor summation convention of Einstein is adopted herein with summation ranging from 1 to n.

The loading λ and the parameter η are the main control parameters for static and dynamic bifurcations as well as for the stability of equilibria and limit cycles. The masses and damping as possible control parameters are discussed too. Dynamic bifurcation is defined as a sudden qualitative change of the system response occurring at a certain value of a smoothly varying control parameter. From a view point of topology, dynamic bifurcations correspond to those values of a control parameter for which the response of the system becomes structurally unstable (Andronov and Pontryagin, 1937); namely the phase portrait is changed to a topologically nonequivalent portrait by a smooth change of the control parameter. It is also assumed that bifurcation points (static or dynamic) lie on a trivial precritical equilibrium path.

The inclusion of damping, in addition to geometrical nonlinearities, allows a more precise modelling of a real structural system. Internal friction, in the most general sense, has as a consequence the existence of an attractor; that is, the existence of an asymptotic limit of the solutions (as $t \to \infty$) such that the initial conditions (i.e. the point of departure) have no direct influence. In mechanics, when friction entails continuous decrease of the energy, the corresponding systems are called for this reason *dissipative*. Many phenomena in nonlinear dynamics are the corollary of the interaction between geometrical nonlinearities and damping, particularly in cases of nongradient systems. The undamped systems can also be treated as the limiting case of systems with vanishing but nonzero damping.

Setting

$$y_1 = q_1, \quad y_2 = q_2, \ldots, y_n = q_n \quad \text{and} \quad y_{n+1} = \dot{q}_1, \quad y_{n+2} = \dot{q}_2, \ldots, y_{2n} = \dot{q}_n, \tag{2}$$

equations (1) for an initially at rest system can be written as follows

$$\left.\begin{array}{l}\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y}\,;\lambda\,;\eta), \quad \mathbf{y} \in E^{2n}, \quad \lambda, \eta \in E \\ \text{subject to } \mathbf{y}(t=0) = 0,\end{array}\right\} \tag{3}$$

where $\mathbf{y} = (y_1, \ldots, y_{2n})^T$ is the state vector in the Euclidean space E^{2n}, being a continuous function of t and λ for fixed η, with T denoting transpose; $\mathbf{Y} = (Y_1, \ldots, Y_{2n})^T$ is a nonlinear vector-function which we assume satisfies the Lipschitz conditions for all t, λ and η, at least in the domain of interest. Due to the above assumptions $y_i(t\,;\lambda\,;\eta)$ belongs to the class of functions C_2. This analysis could be extended to nonautonomous systems since the latter can be transformed to autonomous systems with $\mathbf{y} \in E^{2n+1}$ by letting $y_{2n+1} = t$ and $\dot{y}_{2n+1} = 1$.

According to Cauchy–Lipschitz theorem the solution of the initial-value problem defined by eqn (3) satisfies the integral equation

$$\mathbf{y}(t\,;\lambda\,;\eta) = \mathbf{C} + \int_0^t \mathbf{Y}[\mathbf{y}(s\,;\lambda\,;\eta), s]\, \mathrm{d}s \tag{4}$$

where $C = \mathbf{y}(t=0) = 0$. Clearly, the vector-function $\mathbf{Y}$ is not, in general, integrable. However, an approximate solution of eqn (4) can be obtained via a Taylor's expansion of $\mathbf{Y}$ around a known solution as will be shown below.

The existence of all possible equilibrium states $\mathbf{y}^E$ can be established by setting the L.H.S. of eqn (3) equal to zero, i.e.

$$\mathbf{Y}(\mathbf{y}^E\,;\lambda\,;\eta) = 0 \tag{5}$$

whose equivalent form, if one sets $V_i = \partial U/\partial q_i - Q_i$, is equal to

$$V_i(\mathbf{y}^E\,;\lambda\,;\eta) = 0, \quad (i = 1, \ldots, n). \tag{6}$$

In general, it is not possible to solve the nonlinear initial-value problem of eqns (3); however, a great deal of qualitative information about the local behavior of the solution can be achieved.

Taylor's expansion of eqn (3)

A local analysis refers to the study of the nature of the eigenvalues of the Jacobian matrix evaluated at a known solution $\mathbf{y}^0$ being either an equilibrium (singular) point $\mathbf{y}^E$ or a nonequilibrium (regular) point $\mathbf{y}^R$. For the study of a solution $\mathbf{y}^0$ of eqn (3) we can examine the motion in its neighborhood by superimposing the disturbance (vector) ξ to $\mathbf{y}^0$. Inserting $\mathbf{y} = \mathbf{y}^0 + \xi$ into eqn (3) and using a Taylor's expansion around $\mathbf{y}^0$ we get

$$\dot{\xi} = \delta\mathbf{Y}^0 + \frac{1}{2!}\delta^2\mathbf{Y}^0 + \frac{1}{3!}\delta^3\mathbf{Y}^0 + \cdots \tag{7}$$

where $\delta^m\mathbf{Y}^0 = [\delta^m Y_1^0, \delta^m Y_2^0, \ldots, \delta^m Y_{2n}^0]^T$, $m = 1, 2, \ldots$

$$\delta \mathbf{Y}^0 = \begin{bmatrix} \frac{\partial Y_1^0}{\partial y_1} & \cdots & \frac{\partial Y_1^0}{\partial y_{2n}} \\ \vdots & & \vdots \\ \frac{\partial Y_{2n}^0}{\partial y_1} & \cdots & \frac{\partial Y_{2n}^0}{\partial y_{2n}} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{2n} \end{bmatrix} = \mathbf{Y}_y^0 \boldsymbol{\xi}. \tag{8}$$

Note that

$$\delta^2 Y_i^0 = \left(\xi_1 \frac{\partial}{\partial y_1} + \cdots + \xi_{2n} \frac{\partial}{\partial y_{2n}} \right)^2 Y_i^0,$$

$$\delta^3 Y_i^0 = \left(\xi_1 \frac{\partial}{\partial y_1} + \cdots + \xi_{2n} \frac{\partial}{\partial y_{2n}} \right)^3 Y_i^0, \text{ etc.} \tag{9}$$

where $\mathbf{Y}_y^0 = \mathbf{Y}_y(\mathbf{y}^0; \lambda; \eta) = \partial \mathbf{Y}(\mathbf{y}^0; \lambda; \eta)/\partial \mathbf{y}$ is the Jacobian matrix evaluated at $\mathbf{y}^0$.

According to the Hartman–Grobman theorem and the stable manifold theorem (Perko, 1991) the behavior of solutions of the nonlinear eqn (3) *near* an equilibrium point which is hyperbolic (i.e. none of the Jacobian eigenvalues has zero real part) is qualitatively determined by the behavior of the linear equation $\dot{\boldsymbol{\xi}} = \mathbf{Y}_y^E \boldsymbol{\xi}$ near the origin. If all the eigenvalues of $\mathbf{Y}_y^E$ have negative (positive) real part the equilibrium point is called a sink (source); a hyperbolic equilibrium point is called saddle if the Jacobian matrix $\mathbf{Y}_y^E$ has at least one eigenvalue with a positive real part and one with a negative real part. A hyperbolic equilibrium point $\mathbf{y}^E$ is either asymptotically stable (iff it is a sink) or unstable (iff it is either a source or a saddle). Stable equilibrium points which are not asymptotically stable can occur only at nonhyperbolic equilibrium points. A stable nonhyperbolic equilibrium point is a center iff the Jacobian matrix has either a zero eigenvalue or a pair of complex conjugate, pure imaginary, eigenvalues. However, whether a nonhyperbolic equilibrium point is stable, asymptotically stable or unstable is rather difficult to determine. One of the more useful methods in answering this question is due to Liapunov.

As stated above, all sinks are asymptotically stable. However, not all asymptotically stable equilibrium points are sinks. This is so since a nonhyperbolic equilibrium point can be asymptotically stable (all eigenvalues have zero real parts or some of them have zero real parts and the remaining eigenvalues are zero).

The Jacobian matrix $\mathbf{Y}_y(\mathbf{y}; \lambda; \eta)$ can be written as a block matrix with four submatrices of order $n \times n$; that is (Kounadis, 1993)

$$\mathbf{Y}_y(\mathbf{y}^0; \lambda; \eta) = \begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -[\tilde{V}_{ij}] & -[\tilde{c}_{ij}] \end{bmatrix} \tag{10}$$

where $\mathbf{0}$ and $\mathbf{I}_n$ are the zero and identity matrix; $[\tilde{V}_{ij}] = [a_{ij}]^{-1}[V_{ij}]$ and $[\tilde{c}_{ij}] = [a_{ij}]^{-1}[c_{ij}]$.

The characteristic equation of the Jacobian matrix (10) is

$$|\rho^2 I_n + \rho[\tilde{c}_{ij}] + [\tilde{V}_{ij}]| = 0 \quad \text{or} \quad |\rho^2[a_{ij}] + \rho[c_{ij}] + [V_{ij}]| = 0, \tag{11}$$

which after expansion yields

$$f(\rho) = \rho^{2n} + a_1 \rho^{2n-1} + a_2 \rho^{2n-2} + \cdots + a_{2n-1}\rho + a_{2n} = 0 \tag{12}$$

where

$$a_1 = -\operatorname{tr} \mathbf{Y}_y = \sum_{i=1}^{n} \tilde{c}_{ii} = \sum_{i=1}^{2n} \rho_i, \quad a_{2n} = \det \mathbf{Y}_y = \det[\tilde{V}_{ij}] = \prod_{i=1}^{2n} \rho_i \tag{13}$$

and ρ_i ($i = 1, \ldots, 2n$) are the Jacobian eigenvalues. From eqns (10) and (13) we get that the η buckling loads for which the determinant of the matrix $[\tilde{V}_{ij}]$ vanishes imply also the vanishing of the determinant of the Jacobian matrix. The coefficients a_i ($i = 1, \ldots, n$) can be determined by means of Bôcher's formula (Gantmacher, 1959).

Using eqns (11) and (12) it is found that a_{2n} is a function only of λ and η, while a_1 is a function only of the damping coefficients. Finally, $a_2, a_3, \ldots, a_{2n-1}$ are functions of damping coefficients, λ and η. The characteristic eqn (12) can also be written as follows

$$f(\rho) = \prod_{i=1}^{n} (\rho^2 + B_i\rho + C_i) = 0 \tag{14}$$

$$\left.\begin{array}{l} \text{with roots of each factor} \quad -0.5B_i \pm \sqrt{(0.5B_i)^2 - C_i} \\ \text{where} \quad \sum_{i=1}^{n} B_i = a_1 = \sum_{i=1}^{n} \tilde{c}_{ii}, \quad \text{and} \quad \prod_{i=1}^{n} C_i = a_{2n}(\lambda;\eta) \end{array}\right\}. \tag{15}$$

3. LOCAL BIFURCATIONS

Using a local (linear) analysis one can always establish static bifurcations and in some cases local dynamic bifurcations. Moreover, one can discuss stability of equilibria (excluding the case of critical states) by studying the nature of Jacobian eigenvalues. However, there are dynamic bifurcations (associated with limit cycles) which can be explored by using only a global (nonlinear) dynamic analysis. The determination and stability of global bifurcations is achieved either numerically or with the aid of eqn (4) in which the integrand has been replaced by two or three terms in the expansion (7).

Static bifurcations

For bifurcational systems with trivial fundamental paths, eqn (5) is satisfied by the zero solution, $\mathbf{y}^E \equiv 0$, regardless of the value of λ and η; i.e.

$$\mathbf{Y}(0;\lambda;\eta) = 0. \tag{16}$$

At the critical value, $\lambda = \lambda^c$ (depending on η), the system also exhibits another solution different from zero, $\mathbf{y}^E \neq 0$; namely it displays a bifurcation. The boundary between the regions of existence and nonexistence of adjacent equilibria corresponds to a certain value of η, say $\eta = \eta_0$, which is determined as follows: equating to zero the determinant of the Jacobian matrix evaluated at $\mathbf{y}^E = 0$, we obtain the buckling (divergence) equation

$$a_{2n} = \det \mathbf{Y}_y(0;\lambda^c;\eta) = |\mathbf{Y}_y(0;\lambda^c;\eta)| = 0 \tag{17}$$

which due to eqn (6) yields $\det[\tilde{V}_{ij}] = 0$.

From eqn (17) one can obtain, at least implicitly, the relationship

$$\eta = \eta(\lambda^c). \tag{18}$$

Following the procedure outlined by Kounadis (1983) the extreme value of η, i.e. $\eta = \eta_0$, is determined by the condition

$$\frac{\mathrm{d}n}{\mathrm{d}\lambda} = \eta'(\lambda^c) = 0 \quad \text{or} \quad a_{2n\lambda} = |\mathbf{Y}_{y\lambda}(0;\lambda^c;\eta)| = 0. \tag{19}$$

Let λ_0^c be the smallest positive root (critical load) of eqns (16) and (19) for which $\eta(\lambda_0^c) = \eta_0$. Clearly η_0 defines a bound in the region of existence of adjacent equilibria; if

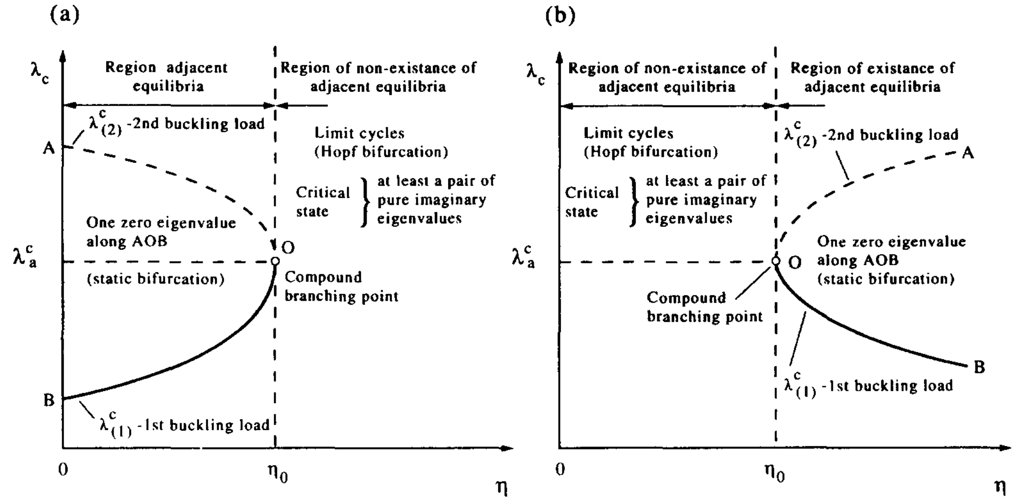

Fig. 1. The compound branching point 0 (λ^c_0, η_0), boundary between the regions of existence and nonexistence of adjacent equilibria. Point 0 in the curve AOB may be either a maximum (a) or a minimum (b).

η_0 is a maximum [Fig. 1(a)] of the function $\eta = \eta(\lambda^c)$, adjacent equilibria exist for $\eta < \eta_0$, while if η_0 is a minimum [Fig. 1(b)] of $\eta = \eta(\lambda^c)$, adjacent equilibria exist for $\eta > \eta_0$; i.e. outside these regions adjacent equilibria do not exist. Since $a_{2n} = a_{2n\lambda} = 0$ at (λ^c_0, η_0) the latter is a coincident (double) point resulting from the coalescence of two consecutive buckling curves η vs λ^c among which the lower corresponds to the smallest (critical) load.

For the precritical states of divergence instability the following observations are worth making regarding eqn (11) or eqn (12). If $[a_{ij}]$, $[c_{ij}]$ and $[V_{ij}]$ are non-negative definite matrices, and either $[a_{ij}]$ or $[V_{ij}]$ is positive definite, then eqn (12) has no roots with positive real parts (Bellman, 1970). Recall that $[a_{ij}]$ is always a positive definite matrix, while $[V_{ij}]$ is positive definite in case of a conservative loading when $\lambda < \lambda^c_{(1)}$, where $\lambda^c_{(1)}$ is the smallest critical (buckling) load. If $[a_{ij}]$, $[c_{ij}]$ and $[V_{ij}]$ are positive definite matrices, all eigenvalues have negative real parts and hence the Jacobian is a stability (or stable) matrix (Kounadis, 1993).

In case of a follower loading, the matrix $[V_{ij}]$ is asymmetric and can always be factored into a product of two symmetric matrices. If one of these symmetric matrices is positive definite, $[V_{ij}]$ is called symmetrizable and the dynamic system associated with it behaves as if it were symmetric (Inman, 1983). For a nondissipative system ($[c_{ij}] = 0$) it was shown that all the Jacobian eigenvalues associated with eqn (11) are purely imaginary if $[V_{ij}]$ is symmetrizable (Kounadis, 1992b). However, if $[c_{ij}] \neq 0$ it is not always possible to find a positive definite transformation matrix which renders both matrices $[c_{ij}]$ and $[V_{ij}]$ symmetrizable; a fact, of course, which does not imply that the precritical states are not asymptotically stable.

Among the more efficient criteria for determining whether all Jacobian eigenvalues have negative real parts are those of Routh–Hurwitz. According to these criteria a necessary condition in order that all eigenvalues have negative real parts is $a_i > 0$ (for all i), while a sufficient condition is all the Routh–Hurwitz determinants Δ_i of even or odd order to be positive. Moreover, a necessary and sufficient condition for all the eigenvalues to lie on the left-hand side of the complex plane is $\Delta_i > 0$ (for all i). In this case the characteristic polynomial $f(\rho) = 0$ has complex conjugate eigenvalues of the form

$$\rho_i = \mu_i + v_i j, \quad (j = \sqrt{-1}) \tag{20}$$

where $\mu_i < 0$ and $v_i > 0$ ($i = 1, \ldots, n$). Due to relation (14) it follows

$$\mu_i = -0.5B_i, \quad v_i = \sqrt{C_i-(0.5B_i)^2} \quad \text{with} \quad B_i > 0, \quad C_i > (0.5B_i)^2. \tag{21}$$

The static (divergence) instability associated with a static bifurcation occurs when the Jacobian or matrix $[V_{ij}]$ becomes singular; i.e. when due to eqns (15) one at least of C_i, say C_k, becomes zero. Then, from the second of eqns (15) it is deduced that the (corresponding to $C_k = 0$) pair of complex conjugate eigenvalues yields a zero eigenvalue $\rho_k = 0$ and a negative eigenvalue (equal to $-B_k$).

Clearly, as λ increases from zero the trivial state is asymptotically stable for $\lambda < \lambda^c_{(1)}$ (where $\lambda^c_{(1)}$ is the smallest critical buckling load) if all the Routh–Hurwitz determinants Δ_i are positive. At the critical (divergence) state corresponding to $\lambda^c_{(1)}$ one Jacobian eigenvalue becomes zero, another (real) negative, and the remaining are complex conjugate with negative real parts. The system may be stable or unstable but locally. Indeed local instability (stability) does not imply necessarily global instability (stability) which may be affected decisively by the presence of nonlinear terms governing the long-term response of the system. The (global) stability or instability can only be established by including higher order terms in the Taylor's expansion (La Salle and Lefschetz, 1961), since stability criteria for linearized models are concerned with local properties rather than global behavior. For λ slightly greater than $\lambda^c_{(1)}$, the determinant of $[V_{ij}]$ (whose elements are linear functions of λ) become negative (in cases of distinct buckling loads). This yields $C_k < 0$, and due to relations (15), (20) and (21), the Jacobian matrix has one positive and one negative eigenvalue, while all the remaining eigenvalues are complex conjugate with negative real parts, namely, the static (divergence) instability takes place when at least one eigenvalue ρ becomes positive after passing through zero at $\lambda = \lambda^c_{(1)}$ (where the Jacobian becomes singular). Thus, for $\lambda < \lambda^c_{(1)}$ the trivial state is a hyperbolic equilibrium point (sink) with all Jacobian eigenvalues complex conjugate with negative real parts. At $\lambda = \lambda^c_{(1)}$ a pair of these complex eigenvalues is transformed to a zero eigenvalue (the trivial state becomes a nonhyperbolic equilibrium point) and to a negative eigenvalue, while for $\lambda > \lambda^c_{(1)}$ the zero eigenvalue becomes positive, another eigenvalue is negative, whereas all the remaining are complex conjugate with negative real parts. Therefore, the trivial state for $\lambda > \lambda^c_{(1)}$ (having a positive eigenvalue) is locally unstable. However, the system may be globally stable exhibiting a point attractor response (after a dynamic jumping to a stable postcritical equilibrium state). The dynamic global stability of the critical and postcritical response depends on the stability of the static bifurcation point.

It is worth observing that the singularity of the Jacobian at $\lambda^c_{(1)}$ (and the transformation of a hyperbolic to a nonhyperbolic equilibrium point) does not imply that the trivial state is associated with a static bifurcation. This is so because the Jacobian becomes also singular when a conjugate pair of eigenvalues coincides at the origin of the ρ-plane and proceeds in opposite directions on the real axis (Huseyin, 1986). In view of this a static bifurcation is characterized by a zero eigenvalue at $\lambda = \lambda^c_{(1)}$ which for λ slightly greater than $\lambda = \lambda^c_{(1)}$ becomes positive. Note also that the double critical point (λ^c_0, η_0) does not satisfy the foregoing properties of a static bifurcation although it is an equilibrium point. This is so because the sign of a_{2n} (being a second degree polynomial of λ^c) does not change for $\lambda > \lambda^c_{(1)}$, remaining always positive for $\lambda \neq \lambda^c_{(1)}$. Thus, for $\eta = \eta_0$ and $\lambda < \lambda^c_{(1)}$ all eigenvalues are complex conjugate with negative real parts. At $\lambda = \lambda^c_{(1)} = \lambda^c_0$ one pair of complex conjugate eigenvalues is transformed to a zero and to a negative eigenvalue (while all the remaining are complex conjugate). At $\lambda > \lambda^c_0$ the last two eigenvalues are transformed to a pair of complex conjugate eigenvalues with negative real parts. The important conclusion is that the double critical point (λ^c_0, η_0) does not behave as an equilibrium point; it is a hybrid or pseudo-equilibrium point. This new finding revises a pertinent aspect for non-gradient but nondissipative (nongeneric) systems (Mandady and Huseyin, 1980).

If $B_k = 0$ and $C_k = 0$ then due to eqn (14) (the remaining eigenvalues are complex conjugate with negative real parts) the Jacobian has a double zero eigenvalue, i.e.

$$a_{2n-1} = a_{2n} = 0. \tag{22}$$

Since $a_{2n} = a_{2n}(\lambda, \eta)$ and $a_{2n-1} = a_{2n-1}(c_{ij}, \lambda, \eta)$ one can determine numerically the extreme values of η (one of which is η_0) defining the region of variation of η (along the length of the common curve η vs λ^c of two consecutive buckling loads coinciding at η_0) for which the Jacobian has a double zero eigenvalue. If the Jacobian is a defective matrix (i.e. being not similar to a diagonal matrix) then the number of linearly independent eigenvectors is $2n-1$ (if the $2n-2$ eigenvalues are distinct). Thus, there corresponds one eigenvector to the double zero eigenvalue occurring along the above two consecutive buckling curves (η vs λ^c). Since one eigenvector corresponds to two consecutive critical (divergence) states (among which the lowest corresponds to the smallest static buckling load) it is reasonable to consider that the corresponding postbuckling modes are not independent of each other [Fig. 2(a, b)]. At the critical states of divergence instability associated with a double zero eigenvalue the system exhibits a limit cycle response. However, regardless of the amount of damping, the system for loads much higher than the smallest critical (divergence) load exhibits a point attractor response in case of a stable postbuckling equilibrium path (Kounadis, 1992a). For much higher loads there is (as stated above) an interaction of postbuckling modes subsequently causing an interaction of static and dynamic bifurcations (Yu and Huseyin, 1988) which can be studied only by using a nonlinear dynamic analysis.

The study of critical (divergence) states with multiple eigenvalues is considerably facilitated by transforming the singular Jacobian matrix to Jordan canonical form. For simplicity we can consider a four-dimensional state space (derived e.g. from higher dimensions after pertinent reduction via the Liapunov–Schmidt, center manifold or spitting lemma technique). In case of a double zero eigenvalue ($a_3 = a_4 = 0$) the characteristic equation, $\rho^2+a_1\rho+a_2 = 0$, yields for small damping [implying $a_2 > (a_1/2)^2$] a pair of complex conjugate roots $\mu+jv$, where $\mu = -0.5a_1$ and $v = (a_2-a_1^2/4)^{1/2}$. At the trivial state $(0;0;\lambda;\eta)$ equation $\dot{\xi} = \mathbf{Y}_y\xi$ via the transformation of variables $\xi = \mathbf{T}\zeta$ yields

$$\dot{\zeta} = \mathbf{J}\zeta \tag{23}$$

where

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{Y}_y\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & v \\ 0 & 0 & -v & \mu \end{bmatrix} \tag{24}$$

with μ and v given above; $\mathbf{Y}_y$ is given in eqn (10), in which $\tilde{V}_{11}\tilde{V}_{22} = \tilde{V}_{21}\tilde{V}_{12}$.

The nonsingular transformation matrix $\mathbf{T}$ with the aid of which the Jacobian matrix is transformed to Jordan form is evaluated as follows.

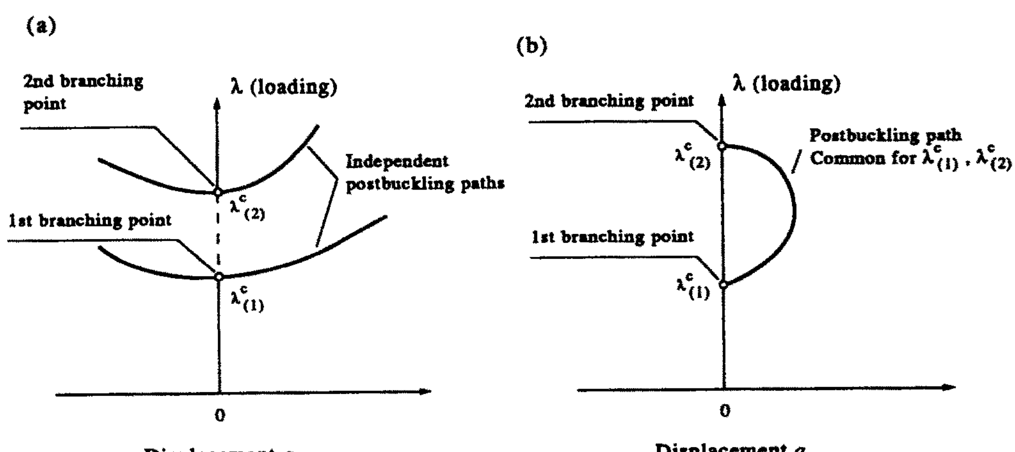

Fig. 2. (a) Independent postbuckling equilibrium paths corresponding to the first and second buckling loads $\lambda^c_{(1)}$ and $\lambda^c_{(2)}$; and (b) one postbuckling path common for both $\lambda^c_{(1)}$ and $\lambda^c_{(2)}$.

The column matrices $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ of the matrix $\mathbf{T}$ are the eigenvectors of the Jacobian $\mathbf{Y}_y(0,0,\lambda,\eta)$ corresponding to the eigenvalues: $\rho_1 = 0$, $\rho_2 = 0$, $\rho_3 = \mu + jv$ and $\rho_4 = \mu - jv$, respectively. With the aid of equations

$$\left.\begin{aligned} \mathbf{Y}_y\mathbf{T}_1 &= \rho_1\mathbf{T}_1 = 0 \\ \mathbf{Y}_y\mathbf{T}_2 &= \rho_2\mathbf{T}_2 + \mathbf{T}_1 = \mathbf{T}_1 \end{aligned}\right\} \tag{25}$$

we find $\mathbf{T}_1 = [1, d, 0, 0]^T$ and $\mathbf{T}_2 = [1, e, 1, d]^T$ where

$$\left.\begin{aligned} d &= -\tilde{V}_{11}/\tilde{V}_{12} = -\tilde{V}_{21}/\tilde{V}_{22} \\ e &= \frac{1}{\tilde{V}_{22}^2}[\tilde{c}_{22}\tilde{V}_{21} - \tilde{V}_{22}(\tilde{V}_{21} + \tilde{c}_{21})] \end{aligned}\right\}. \tag{26}$$

Finally by virtue of equations

$$\left.\begin{aligned} (\mathbf{Y}_y - \mu\mathbf{I})\mathbf{T}_3 + v\mathbf{T}_4 &= 0 \\ (\mathbf{Y}_y - \mu\mathbf{I})\mathbf{T}_4 - v\mathbf{T}_3 &= 0 \end{aligned}\right\} \tag{27}$$

we obtain

$$\mathbf{T}_3 = [1, f, \mu, (\mu f - vg)]^T \quad \text{and} \quad \mathbf{T}_4 = [0, g, v, (vf + \mu g)]^T$$

where

$$\left.\begin{aligned} f &= \frac{1}{D}[(\tilde{V}_{12} + \mu\tilde{c}_{12})(v^2 - \mu^2 - \mu\tilde{c}_{11} - \tilde{V}_{11}) - v^2\tilde{c}_{12}(2\mu + \tilde{c}_{11})] \\ g &= -\frac{1}{D}[(\tilde{V}_{12} + \mu\tilde{c}_{12})(2v\mu + v\tilde{c}_{11}) + v\tilde{c}_{12}(v^2 - \mu^2 - \mu\tilde{c}_{11} - \tilde{V}_{11})] \\ D &= (\tilde{V}_{12} + \mu\tilde{c}_{12})^2 + v^2\tilde{c}_{12}^2 \end{aligned}\right\} \tag{28}$$

The solution of eqn (23) is

$$\zeta(t) = e^{t\mathbf{J}}\zeta(0) \tag{29}$$

where

$$e^{t\mathbf{J}} = \begin{bmatrix} e^{t\mathbf{J}_0} & 0 \\ 0 & e^{t\mathbf{J}_1} \end{bmatrix}, \quad \mathbf{J}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{J}_1 = \begin{bmatrix} \mu & v \\ -v & \mu \end{bmatrix}. \tag{30}$$

Given that

$$e^{t\mathbf{J}_0} = I + t\mathbf{J}_0, \quad e^{t\mathbf{J}_1} = e^{\mu t}\begin{bmatrix} \cos vt & \sin vt \\ -\sin vt & \cos vt \end{bmatrix} \tag{31}$$

eqn (29), due to eqns (30) and (31); becomes

$$\begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \\ \zeta_3(t) \\ \zeta_4(t) \end{bmatrix} = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\mu t}\cos vt & e^{\mu t}\sin vt \\ 0 & 0 & -e^{\mu t}\sin vt & e^{\mu t}\cos vt \end{bmatrix} \begin{bmatrix} \zeta_1(0) \\ \zeta_2(0) \\ \zeta_3(0) \\ \zeta_4(0) \end{bmatrix}. \tag{32}$$

The Jacobian matrix $\mathbf{Y}_y(0,0,\lambda,\eta)$ with a double zero eigenvalue, being defective (since it is not similar to a diagonal matrix) and nonderogatory (the double zero eigenvalue is associated with one Jordan block), has three linearly independent eigenvectors. Equation

(23), due to eqn (32), implies a divergent motion and hence the critical (trivial) state is unstable. It should be clarified that while the linearized model is unstable, the original nonlinear one might be stable since its long-term behavior depends on higher order terms in the Taylor's expansion (see next section).

Dynamic bifurcations

As stated above, via a local dynamic analysis one can predict in some cases dynamic (local) bifurcations. A common type of dynamic bifurcation is the Hopf bifurcation, occurring when the characteristic equation $f(\rho) = 0$ evaluated at $\mathbf{y}^E = 0$ has at least one pair of pure imaginary eigenvalues, while the rest of the eigenvalues are complex conjugate with negative real parts. The critical behavior is thus associated with the vanishing of the real part of at least one pair of complex eigenvalues. This, due to relation (15), implies that one of B_i becomes zero, say $B_j = 0$ and hence the purely imaginary eigenvalues are $\pm\sqrt{C_j}$. Since all B_i $(i = 1, \ldots, n)$, being functions of λ, are assumed negative at the precritical states and at the critical state at least one B_i becomes zero, $B_j = 0$, for a load λ slightly greater than the critical one (since $\mathrm{d}B_j/\mathrm{d}\lambda \neq 0$), B_j becomes positive. This implies instability of the trivial state. The system exhibits an oscillatory motion (limit cycles) which may be (globally) stable or unstable (Nemytskii and Stepanov, 1960). The critical state associated with a Hopf bifurcation is established by setting equal to zero the Routh–Hurwitz determinant Δ_{2n-1}, which due to Orlando's formula (Gantmacher, 1959) is given by

$$\Delta_{2n-1} = (-1)^{n(2n-1)} \prod_{i<j}^{1,\ldots,2n} (\rho_i + \rho_j) \tag{33}$$

which implies that the sum of at least two eigenvalues of $f(\rho) = 0$ is zero.

Since attention is focused on the existence of dynamic bifurcations in the region of adjacent equilibria [Fig. (1a, b)] occurring prior to divergence (i.e. $a_{2n} > 0$) the cases of a pair of opposite eigenvalues or of a double zero eigenvalue are excluded. The last case will be considered thereafter as a special case. Therefore, eqn (33) implies that $f(\rho) = 0$ has at least one pair of pure imaginary eigenvalues.

Clearly, eqn (33), after expansion of Δ_{2n-1}, leads to an algebraic polynomial of nth degree with respect to λ. If the minimum λ, being equal to λ_{cr}, is such that $\lambda_{cr} < \lambda^c_{(1)}$, where $\lambda^c_{(1)}$ is the minimum critical (divergence) buckling load (resulting from $a_{2n} = 0$), then dynamic instability takes place prior to static (divergence) instability. The occurrence of such a phenomenon, very important for structural design purposes, contradicts the widely accepted practice of establishing divergence buckling loads of nonconservative systems under follower loads using static analyses. From the structure of the Routh–Hurwitz determinant it is apparent that eqn (33) holds also when $a_{2n-1} = a_{2n} = 0$. In such a case the critical dynamic load (associated with a Hopf bifurcation) coincides with a critical static (divergence) buckling load (associated with a double zero eigenvalue); i.e. $\lambda_{cr} = \lambda^c_{(1)}$. Hence, a dynamic Hopf bifurcation with $\lambda_{cr} < \lambda^c_{(1)}$ may occur for suitable values of damping coefficients in a small region of adjacent equilibria in the neighborhood of the double critical (divergence) point, being defined by $\eta_0 \leqslant \eta \leqslant \eta_1$, where η_1 is the upper bound of η up to which the Jacobian has a double zero eigenvalue.

Considering the variation of λ^c vs η to the left and right of the point (λ^c_0, η_0), being the boundary between divergence and flutter (oscillatory) instability, the following observation is worth making: between divergence (static) and flutter (dynamic) critical load there is always a discontinuity at the point (λ^c_0, η_0) if the flutter load obtained from eqn (33) corresponds to $a_{2n} \neq 0$. This load may be higher or lower than the divergence buckling load depending on the damping coefficients. This contradicts the corresponding finding of the classical (linear) analysis according to which the flutter critical load is always greater than the corresponding divergence buckling load. On the other hand if $a_{2n} = 0$ then eqn (33) also yields $a_{2n-1} = 0$. Hence, in case of a double zero eigenvalue at (λ^c_0, η_0) there is no discontinuity between the corresponding divergence and flutter critical loads. This finding (occurring for suitable damping coefficients) also contradicts the well-known result of the classical (linear) analysis implying that there is always a discontinuity between the divergence and flutter critical loads at (λ_0^c, η_0).

4. GLOBAL BIFURCATIONS

Global bifurcations can be established by using only a nonlinear (global) analysis. For instance, dynamic bifurcations with trajectories passing through (or approaching) saddle points or cases where closed orbits become nonhyperbolic (at least one characteristic multiplier has unit modulus) can be detected only by using a global (nonlinear) analysis (Peixoto, 1959). Such an analysis is also the only safe way for exploring chaotic or chaos-like (Kounadis, 1991) phenomena appearing sometimes at a large time.

Are there dynamic global bifurcations occurring before static (divergence) buckling? To the knowledge of the author there are no pertinent criteria answering this important question as in the case of a Hopf (local) bifurcation presented above.

The importance of a global dynamic analysis lies mainly in the fact that by this analysis one can readily establish the stability of equilibria and limit cycles in the precritical, critical and postcritical stages. The stability of equilibria is the simplest case if dynamic instability does not occur prior to static (divergence) buckling. The stability of the critical and precritical state depends on the nature of the static stability of the bifurcation point. An interesting treatment on this topic was presented by Plaut (1976). If the nongradient damped system displays a stable branching point, it exhibits a point attractor (as in the case of conservative damped systems). If the branching point is unstable the system is subjected to dynamic buckling (i.e. a very small change in the load produces a large change in the response).

However, the stability of limit cycles for local (e.g. Hopf) or global bifurcations can be accomplished only via a nonlinear (global) analysis; e.g. via perturbation schemes, the average or the intrinsic harmonic balancing technique (Atadan and Huseyin, 1985) or approximate analytic methods (Kounadis, 1992c). These methods are usually adjusted for problems in two dimensions obtained from problems of higher order after reduction of their dimension via the Liapunov–Schmidt technique, and mainly by the local techniques of the center manifold (Carr, 1981), of the normal forms (Perko, 1991) and the splitting lemma (Gilmore, 1981). The global analysis herein is performed through numerical simulation (via a Runge–Kutta numerical scheme) or by using the above approximate analytic technique.

5. NUMERICAL EXAMPLE

Consider, as example, the Ziegler two-degree-of-freedom nonlinear damped model shown in Fig. 3. The model, under a partial follower force at its tip with η the nonconservativeness parameter, is governed by (Kounadis, 1992a)

$$(1+m)\ddot{\theta}_1+\ddot{\theta}_2\cos(\theta_1-\theta_2)+\dot{\theta}_2^2\sin(\theta_1-\theta_2)+c_{11}\dot{\theta}_1+c_{12}\dot{\theta}_2+\frac{\partial V}{\partial\theta_1}=0$$

$$\ddot{\theta}_2+\ddot{\theta}_1\cos(\theta_1-\theta_2)-\dot{\theta}_1^2\sin(\theta_1-\theta_2)+c_{12}\dot{\theta}_1+c_{22}\dot{\theta}_2+\frac{\partial V}{\partial\theta_2}=0 \qquad (34)$$

where $c_{11}=b_1+b_2$, $c_{12}=-c_{22}=-b_2$ with b_1 and b_2 the linear viscous coefficients of the two springs; $m=m_1/\mathrm{m}_2$ is the mass ratio, while

$$\frac{\partial V}{\partial\theta_1}=2\theta_1-\theta_2+\delta_1\theta_1^2-\delta_2(\theta_1-\theta_2)^2+\gamma_1\theta_1^3+\gamma_2(\theta_1-\theta_2)^3-\lambda\sin[\theta_1+(\eta-1)\theta_2]$$

$$\frac{\partial V}{\partial\theta_2}=-\theta_1+\theta_2+\delta_2(\theta_1-\theta_2)^2-\gamma_2(\theta_1-\theta_2)^3-\lambda\sin\eta\theta_2 \qquad (35)$$

with η varying from $\eta=0$ (tangential load) to $\eta=1$ (conservative load).

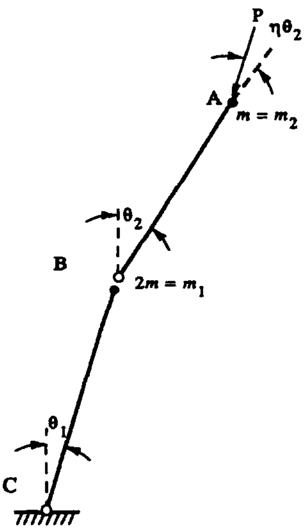

Fig. 3. Ziegler's (1968) two-degree-of-freedom model under partial follower load.

For $|\delta_1|+|\delta_2| \neq 0$ and $\gamma_1 = \gamma_2 = 0$ (quadratic model) the system exhibits an asymmetric bifurcation point provided that $\delta_1+\delta_2(1-\lambda_c)^3 \neq 0$ (Kounadis, 1994), while for $|\gamma_1|+|\gamma_2| \neq 0$ and $\delta_1 = \delta_2 = 0$ (cubic model) it exhibits a symmetric bifurcation point, whose stability depends on γ_1 and γ_2. Setting according to eqns (2)

$$y_1 = \theta_1, \quad y_2 = \theta_2, \quad y_3 = \dot{\theta}_1, \quad y_4 = \dot{\theta}_2 \tag{36}$$

eqns (35) are written as follows

$$\dot{y}_i = Y_i(y_1, y_2, y_3, y_4; \lambda; \eta) \quad (i = 1, \ldots, 4) \tag{37}$$

where $Y_1 = y_3$, $Y_2 = y_4$

$$Y_3 = \frac{1}{m+\sin^2(y_1-y_2)} \left\{ b_2[1+\cos(y_1-y_2)]y_4 - [b_1+b_2+b_2\cos(y_1-y_2)]y_3 \right.$$
$$\left. - \tfrac{1}{2}y_3^2 \sin 2(y_1-y_2) - y_4^2 \sin(y_1-y_2) - \frac{\partial V}{\partial y_1} + \frac{\partial V}{\partial y_2}\cos(y_1-y_2) \right\}$$

$$Y_4 = \frac{1}{m+\sin^2(y_1-y_2)} \left\{ [(1+m)b_2 + (b_1+b_2)\cos(y_1-y_2)]y_3 - b_2[1+m+\cos(y_1-y_2)]y_4 \right.$$
$$\left. + (1+m)y_3^2 \sin(y_1-y_2) + \frac{y_4^3}{2}\sin 2(y_1-y_2) - (1+m)\frac{\partial V}{\partial y_2} + \frac{\partial V}{\partial y_1}\cos(y_1-y_2) \right\}. \tag{38}$$

The three terms on the R.H.S. of eqn (7) evaluated at the trivial equilibrium state ($\theta_1 = \theta_2 = 0$) are

$$\delta Y^E = Y_y^E \boldsymbol{\xi} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{\lambda-3}{m} & \dfrac{2-\lambda}{m} & -\dfrac{b_1+2b_2}{m} & \dfrac{2b_2}{m} \\ \dfrac{m+3-\lambda}{m} & \dfrac{\lambda(1+mn)-m-2}{m} & \dfrac{b_1+(m+2)b_2}{m} & \dfrac{-b_2(m+2)}{m} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} \tag{39}$$

$$\frac{1}{2!}\delta^2 Y^E + \frac{1}{3!}\delta^3 Y^E = [0, 0, \tfrac{1}{2}\delta^2 Y_3^E + \tfrac{1}{6}\delta^3 Y_3, \tfrac{1}{2}\delta^2 Y_4 + \tfrac{1}{6}\delta^3 Y_4]^T \tag{40}$$

where

$$\tfrac{1}{2}\delta^2 Y_3 = \frac{1}{m}[(2\delta_2-\delta_1)\xi_1^2 + 2\delta_2\xi_2^2 - 4\delta_2\xi_1\xi_2]$$

$$\tfrac{1}{2}\delta^2 Y_4^E = \frac{1}{m}\{[\delta_1-(m+2)\delta_2]\xi_1^2 - (m+2)\delta_2\xi_2^2 + 2(m+2)\delta_2\xi_1\xi_2\}$$

$$\begin{aligned}\tfrac{1}{6}\delta^3 Y_3^E = & -\frac{b_2(m+4)}{2m^2}(\xi_1-\xi_2)^2\xi_4 + \frac{[2b_1+b_2(m+4)]}{2m^2}(\xi_1-\xi_2)^2\xi_3 - \frac{(\xi_1-\xi_2)}{m}\xi_4^2 \\ & -\frac{(\xi_1-\xi_2)}{m}\xi_3^2 - \frac{1}{m}[\gamma_1\xi_1^3 + 2\gamma_2(\xi_1-\xi_2)^2] + \frac{\lambda^c}{6m}\{[\eta^3-(\eta-1)^3]\xi_2^3 - \xi_1^3 \\ & -3(\eta-1)\xi_1^2\xi_2 - 3(\eta-1)^2\xi_2^2\xi_1\} + \frac{(\xi_1-\xi_2)^2}{m^2}\left\{\left(\frac{m+6}{2}-\lambda^c\right)\xi_1\right. \\ & \left. -\left[\frac{m+4}{2}-\lambda^c\left(1+\frac{m\eta}{2}\right)\right]\xi_2\right\}\end{aligned}$$

$$\begin{aligned}\tfrac{1}{6}\delta^3 Y_4^E = & \frac{b_2(4+3m)}{2m^2}(\xi_1-\xi_2)^2\xi_4 - \frac{(m+2)b_1+(4+3m)b_2}{2m^2}(\xi_1-\xi_2)^2\xi_3 + \frac{(\xi_1-\xi_2)}{m}\xi_4^2 \\ & + \frac{(m+1)}{m}(\xi_1-\xi_2)\xi_3^2 + \frac{1}{m}[\gamma_1\xi_1^3 + (m+2)\gamma_2(\xi_1-\xi_2)^2] + \frac{\lambda^c}{6m}\{[\xi_1+(\eta-1)\xi_2]^3 \\ & -(m+1)\eta^3\xi_2^3\} + \frac{(\xi_1-\xi_2)^2}{2m^2}\{[(m+2)\lambda^c-4m-6]\xi_1 - [(m\eta+m+2)\lambda^c-3m-4]\xi_2\}.\end{aligned} \tag{41}$$

An iteration scheme is successfully employed (Kounadis, 1992c) by replacing the R.H.S. of eqn (3) by its three term Taylor's expansion given in eqns (39), (40) and (41). By this analytic approximate technique or via numerical simulation we establish the stability of critical states associated either with a double zero eigenvalue or with limit cycles.

The characteristic equation of the Jacobian matrix of eqn (39) is

$$\rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4 = 0 \tag{42}$$

where

$$\left.\begin{aligned} a_1 &= \frac{1}{m}[b_1+(m+4)b_2], \quad a_2 = \frac{1}{m}[b_1b_2+m+5-\lambda(2+m\eta)] \\ a_3 &= \frac{1}{m}[b_1(1-\lambda\eta)+b_2(1-2\lambda\eta)], \quad a_4 = \frac{1}{m}(\eta\lambda^2-3\lambda\eta+1) \end{aligned}\right\}. \tag{43}$$

Since the damping coefficients b_1 and b_2 are always (small) positive quantities, $a_1 > 0$. Application of relation (15) gives

$$\left.\begin{aligned} &\rho_{1,2} = -\frac{B_1}{2} \pm \sqrt{\frac{B_1^2}{4} - C_1}, \quad \rho_{3,4} = -\frac{B_2}{2} \pm \sqrt{\frac{B_2^2}{4} - C_2} \\ &\text{where} \\ &B_1+B_2 = a_1 > 0, \quad C_1+C_2+B_1B_2 = a_2, \quad C_1B_2+C_2B_1 = a_3, \quad C_1C_2 = a_4 \end{aligned}\right\}. \tag{44}$$

The boundary between the regions of existence and nonexistence of adjacent equilibria is determined by solving the system of eqns (17) and (19); i.e. $a_4 = \mathrm{d}a_4/\mathrm{d}\lambda = 0$ from which we get the double (coincident) critical point

$$\eta_0 = \tfrac{4}{9}, \quad \lambda_0^c = \tfrac{3}{2}. \tag{45}$$

Namely for $4/9 < \eta < 1$ the model displays a divergence instability, while for $0 < \eta < 4/9$ there are no adjacent equilibria and the model exhibits a limit cycle (stable or unstable) response. The latter region is also known as region of flutter (dynamic) instability. The first and second static (divergence) buckling loads $\lambda_{(1)}^c$ and $\lambda_{(2)}^c$ are obtained through the equation $a_4 = 0$, which yields

$$\lambda_{1,2}^c = \frac{1}{2}\left(3 \pm \sqrt{9-\frac{4}{\eta}}\right) \quad (4/9 < \eta < 1). \tag{46}$$

The dynamic (flutter) critical load λ_{cr} is obtained through the equation

$$\Delta_3 = (a_1a_2-a_3)a_3-a_1^2a_4 = 0. \tag{47}$$

Note that eqn (47) can also be obtained by inserting into eqn (42) $\rho = \pm iv$ [v = real, $i = (-1)^{1/2}$] and thereafter eliminating v. Equation (47) due to eqns (43) leads to an algebraic equation of second degree in λ in the form

$$F(\lambda, \eta; m; b_1; b_2) = 0. \tag{48}$$

Although eqn (48) holds for $0 < \eta < 4/9$ the important question which will be discussed below is whether this equation is also valid in some region of adjacent equilibria. From eqn (48) it is clear that λ_{cr} depends on both mass ratio and damping, contrary to the case of static (divergence) instability where these parameters have no effect on the static buckling load λ^c. According to the local analysis the necessary and sufficient conditions for the stability of precritical states is $a_i > 0$ ($i = 1, \ldots, 4$) and $\Delta_3 > 0$. For a double zero eigenvalue we have $a_3 = a_4 = 0$ [see eqns (22)] which due to (43) yields

$$\lambda^c = \frac{1+2b}{1+b}, \quad \eta = \frac{(b+1)^2}{(b+2)(2b+1)} \tag{49}$$

where $b = b_1/b_2$. Apparently for $b \to 0$ we get $\lambda^c = 1$, while for $b \to \infty$ we obtain $\lambda^c = 2$. In both cases $\eta = 1/2$. Note that λ coincides with the first buckling load $\lambda_{(1)}^c$ if $b < 1$, and with

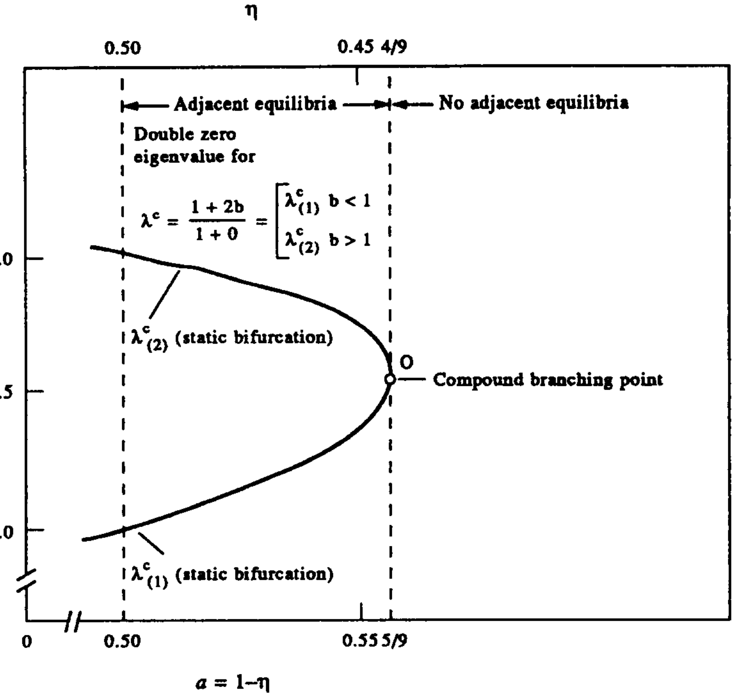

Fig. 4. The small region of adjacent equilibria ($4/9 < \eta < 0.5$) in the neighborhood of the point 0, where a double zero eigenvalue may occur for a suitable damping ratio b.

the second buckling load $\lambda^c_{(2)}$ if $b > 1$ (see Fig. 4). Namely, in the small region of adjacent equilibria defined by $4/9 < \eta < 1/2$ (in the neighborhood of the double critical point) the Jacobian has a double zero eigenvalue along the curve of the first and second static (divergence) buckling load. It can be shown that regardless of the value of η (within the above region) the Jacobian matrix cannot be put into diagonal form; hence it is a defective matrix which implies that one eigenvector corresponds to both (zero) eigenvalues. For instance for a Hookean material ($\delta_1 = \delta_2 = \gamma_1 = \gamma_2 = 0$) with $\eta = 0.48$ (implying $\lambda^c_{(1)} = 1.09176$ and $\lambda^c_{(2)} = 1.90285$) using eqn (49) for $\lambda^c = \lambda^c_{(1)}$ we get $b = b_1/b_2 = 0.101$. For structural (small) damping (e.g. b_2, $b_1 < 0.08$) and mass ratio $m > 0.01$ it is deduced that $a_1^2/4 < a_2$ which implies that the Jacobian matrix in addition to the double zero eigenvalue has a pair of complex conjugate roots $\rho_{3,4} = \mu + iv$, where $\mu = -a_1/2 < 0$ and $v = (a_2 - a_1^2/4)^{1/2}$. Then, either $C_1 = B_1 = 0$ or $C_2 = B_2 = 0$. By virtue of the transformation matrix **T** with columns given in eqns (25–28) we find that the matrix $\mathbf{T}^{-1}\mathbf{Y}_y\mathbf{T}$ has the form of the Jordan matrix **J** given in eqn (24). In this region ($4/9 < \eta < 1/2$), where one eigenvector corresponds to a double zero eigenvalue, there exists only one postbuckling path passing through the first and second branching point as shown in Fig. 5. Above a certain level of the load λ (higher than $\lambda^c_{(1)}$ but lower than $\lambda^c_{(2)}$) the model does not exhibit a point attractor (as it does for $\lambda^c_{(1)} < \lambda < \overset{*}{\lambda}$); it experiences a stable limit cycle (Fig. 6). The stability of the postcritical limit cycle response has been established numerically via the analytic approximate technique mentioned above [eqns (4), (7) and (40)] as well as by using the center manifold technique (Jin and Matzuzaki, 1988).

Note that for $b = b_1/b_2 = 1$ the double critical point (λ^c_0, η_0), due to eqns (43) and (45), is associated with a double zero eigenvalue ($a_3 = a_4 = 0$). Given that at this point

$$a_4 = \frac{4}{9m}(\lambda - \tfrac{3}{2})^2 \tag{50}$$

it is clear that a_4 remains always positive. Another important finding is that $a_3 = a_4 = 0$

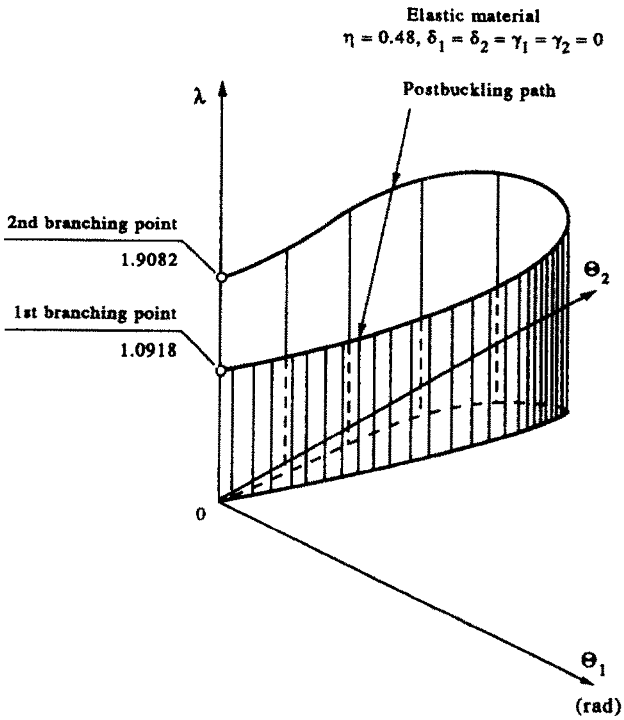

Fig. 5. One postbuckling equilibrium path (passing through the first and second branching point) for $\eta = 0.48 \in [4/9, 0.5]$.

implies also $\Delta_3 = 0$ (Hopf bifurcation); that is for $b = 1$ there is no discontinuity of the critical load at this point as we pass from static (divergence) to dynamic (flutter) instability. This result holds for vanishing (but nonzero) damping with $b_1 = b_2$; that is when the model is practically undamped. This contradicts the well-known result of the classical (linear) analysis according to which there is always a discontinuity in the critical load of the undamped model at (λ_0^c, η_0). Hence, this point does not have the characteristics of the divergence instability (one eigenvalue has positive real part for $\lambda > \lambda_{(1)}^c$), being also a dynamic bifurcation. Moreover, note that all static (divergence) critical states with a double zero eigenvalue ($a_3 = a_4 = 0$) are also Hopf bifurcations since for these equilibrium states $\Delta_3 = 0$. This Hopf bifurcation is independent of the mass ratio.

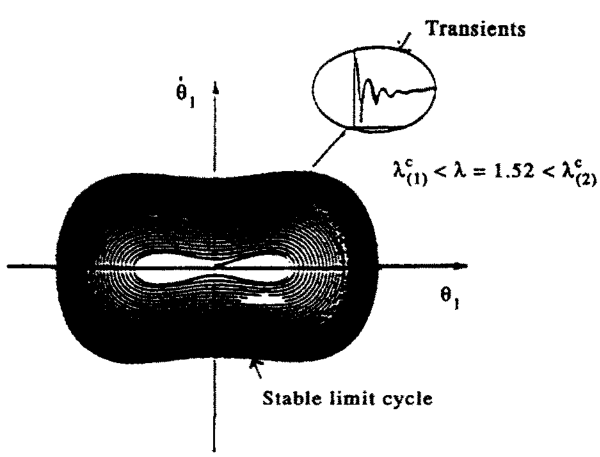

Fig. 6. Phase plane $(\dot{\theta}_1, \theta_1)$ of a damped $(b_1 = b_2 = 0.1)$ system with $\eta = 0.445$, $\gamma_1 = \gamma_2 = \gamma = 0$ and $\delta_1 = \delta_2 = \delta = 0$.

We have established above that two types of dynamic bifurcations may occur in the small region of adjacent equilibria defined by $4/9 < \eta < 1/2$, where there exists a double zero eigenvalue. The first type of dynamic bifurcation occurs for a certain $\lambda = \overset{*}{\lambda} > \lambda^c_{(1)}$ (associated with stable limit cycles), while the second one is a dynamic bifurcation occurring at $\lambda = \lambda^c_{(1)}$ (or $\lambda^c_{(2)}$).

Equation (47) yields the following equation of Hopf bifurcations

$$\{(b+m+4)[b_1b_2+m+5-\lambda(2+m\eta)]-m[b(1-\lambda\eta)+1-2\lambda\eta]\}[b(1-\lambda\eta)+1-2\lambda\eta] \\ -(b+m+4)^2(\eta\lambda^2-3\lambda\eta+1)=0, \quad (b=b_1/b_2). \quad (51)$$

Equation (51) for $b = 1$ and $m = 2$ leads to

$$\eta(24\eta-7)\lambda^2-[28+\eta(4+21b_1^2)]\lambda+41+14b_1^2=0 \quad (52)$$

which for $\eta = 0$ yields

$$\lambda_{cr} = \tfrac{41}{28}+\tfrac{1}{2}b_1^2 \quad \text{or for} \quad b_1 \to 0 \quad \lambda_{cr} = \tfrac{41}{28}. \quad (53)$$

For $\eta \neq 0$ and vanishing damping eqn (51) yields

$$\lambda_{cr} = \frac{14+2\eta-\sqrt{196+343-980\eta^2}}{(24\eta-7)\eta} \quad (54)$$

which is valid for $0 < \eta < 0.65523$ since η varies between 0 and 1. Figure 7 shows the variation of λ_{cr} vs $\eta(>0)$ for $m = 2$ and $b = 2$, 1, 0.4 and zero. Note that all curves λ_{cr} vs η which correspond to $m = 2$ and various values of the damping ratio b pass through the point ($\lambda = 2$, $\eta = 0.625$). Indeed, eqn (51) for $m = 2$, being equal to

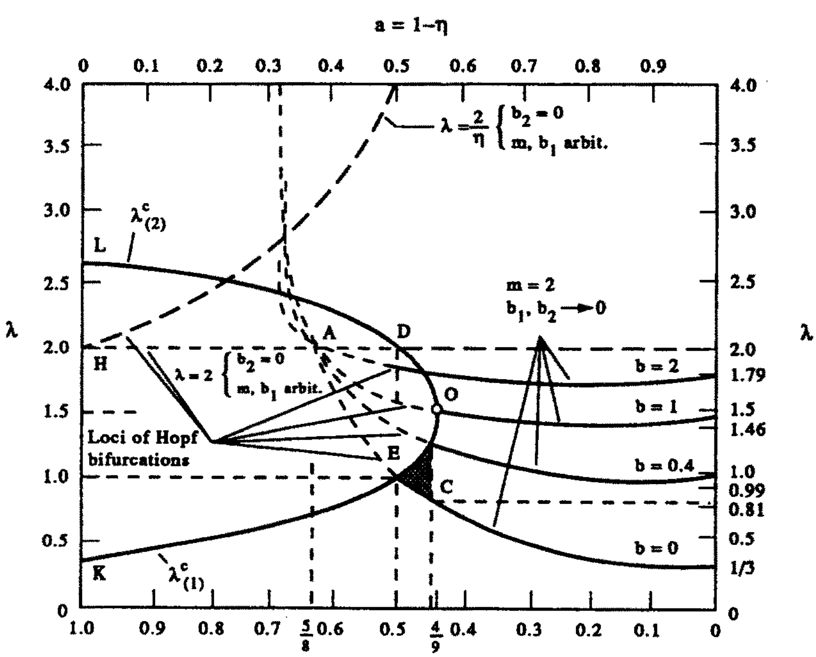

Fig. 7. Loci of Hopf bifurcations of Ziegler's (1968) model with vanishing damping for $m = 2$, $b = 2$, 1, 0.4 and 0.

$$(\lambda\eta-2)(\lambda-2)b^2+(33-14\lambda-22\lambda\eta+4\lambda^2\eta+8\lambda^2\eta^2)b+4+20\lambda\eta-12\lambda+16\lambda^2\eta^2-12\eta\lambda^2=0 \tag{55}$$

is satisfied for $\lambda = 2$ and $\eta = 0.625$ regardless of the damping ratio b. The last case ($b_1 = 0$, b_2 = arbitrary) gives the minimum λ_{cr} since this load decreases with decreasing b. From these curves, being the loci of Hopf (dynamic) bifurcations, the following very important (for structural design purposes) conclusions are drawn. For vanishing damping, $b_1, b_2 \to 0$ (practically undamped system), there exists a region of existence of adjacent equilibria (in the neighborhood of the double critical point) where dynamic instability may occur *prior* to static (divergence) buckling. The maximum width of this region (starting from η_0) corresponds to the case $b = 0$ ($b_1 = 0$, b_2 = arbitrary). Due to eqn (51) and $a_4 = 0$ we find that all curves λ_{cr} vs η corresponding to various m intersect the curve $\lambda^c_{(1)}$ vs η at the point $\lambda^c_{(1)} = 1$ and $\eta_1 = 0.5$ (Fig. 8). Namely, in this region of adjacent equilibria ($4/9 < \eta < 0.5$) the static stability analyses *fail* to predict the actual critical load when $b = 0$ and $b_2 \to 0$ (practically undamped system). However, this model loses its stability through divergence if the corresponding to the ratio b curve (51) intersects the curve $\lambda^c_{(1)}$ vs η at $\lambda > 1$. Therefore, if $b \geqslant 1$ the static criterion is always applicable for $\eta \geqslant 4/9$, while for $b < 1$ and $4/9 \leqslant \eta \leqslant 0.5$ this criterion may fail to predict the actual critical load.

One can further discuss the effect of the mass ratio m on the curve λ_{cr} vs $\eta \in [4/9, 0.5]$ for the case $b = 0$, $b_2 \to 0$. Then one can obtain

$$\lambda_{cr} = \frac{2}{m+4-2m\eta-4\eta}. \tag{56}$$

From the corresponding to various m curves we observe that, as η varies from $\eta_0 = 4/9$ to $\eta_1 = 0.5$, the maximum λ_{cr} is 1 (common for all curves), while the minimum λ_{cr} (occurring at $\eta_0 = 4/9$) decreases as m increases; for $m \to \infty$ we get min $\lambda_{cr} \to 0$ (see Fig. 8). In

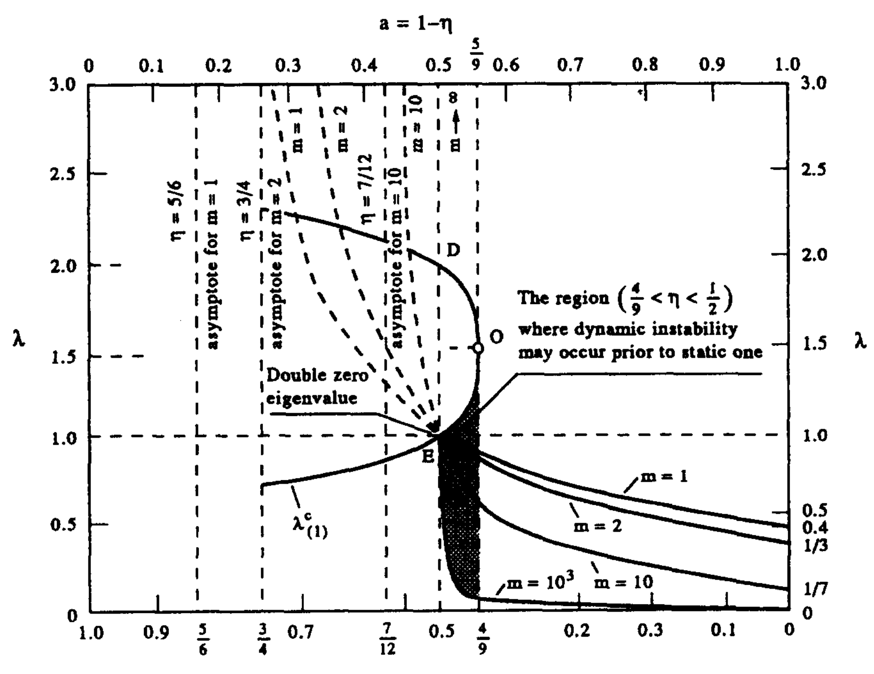

Fig. 8. Loci of Hopf bifurcations of Ziegler's (1968) model for various mass ratios m and $b = 0$, $b_2 \to 0$ (vanishing damping).

conclusion, the region of adjacent equilibria below the curve $\lambda^c_{(1)}$ vs $\eta \in [4/9, 0.5]$ is a domain of dynamic instability (related to Hopf bifurcations) occurring before static instability.

Note also that the dynamic critical load λ_{cr} depends on the mass and damping ratios m and b (even for the case b_1, $b_2 \to 0$), although the model is bifurcational with trivial fundamental path. From eqn (46) it is clear that these parameters have no effect on the static (divergence) buckling load λ^c.

From Fig. 9 one can also see Hopf bifurcations in the region $0 \leqslant \eta < 1$ when either of b_1 or b_2 is zero. It is important to note that among these Hopf bifurcations, the ones physically acceptable are those for $0 \leqslant \eta < 0.5$ since for $\eta > 0.5$ divergence instability occurs for lower loads.

By virtue of eqn (55) one can find for given η and varying b the maximum load λ_{cr}. This load corresponding to a certain damping ratio $b = b_{cr}$ is obtained by the condition $d\lambda/db = 0$ $(d^2\lambda(b_{cr})/db^2 < 0)$ which yields

$$64\eta^2(1+\eta^2)\lambda^4 - 32\eta(7\eta^2+14\eta+5)\lambda^3 + 4(229\eta^2+280\eta+25)\lambda^2 - 20(87\eta+35)\lambda + 1025 = 0. \quad (57)$$

It can be shown that the minimum positive root which satisfies eqn (57) is

$$\lambda_{cr} = \frac{7+\eta-\sqrt{8+14\eta-40\eta^2}}{2(1+\eta^2)}. \quad (58)$$

Note that the latter critical load is identical with that of the corresponding undamped linear system (Herrmann and Bungay, 1964). Hence it established the important finding that the critical load of the damped system has, as upper bound, the corresponding load of

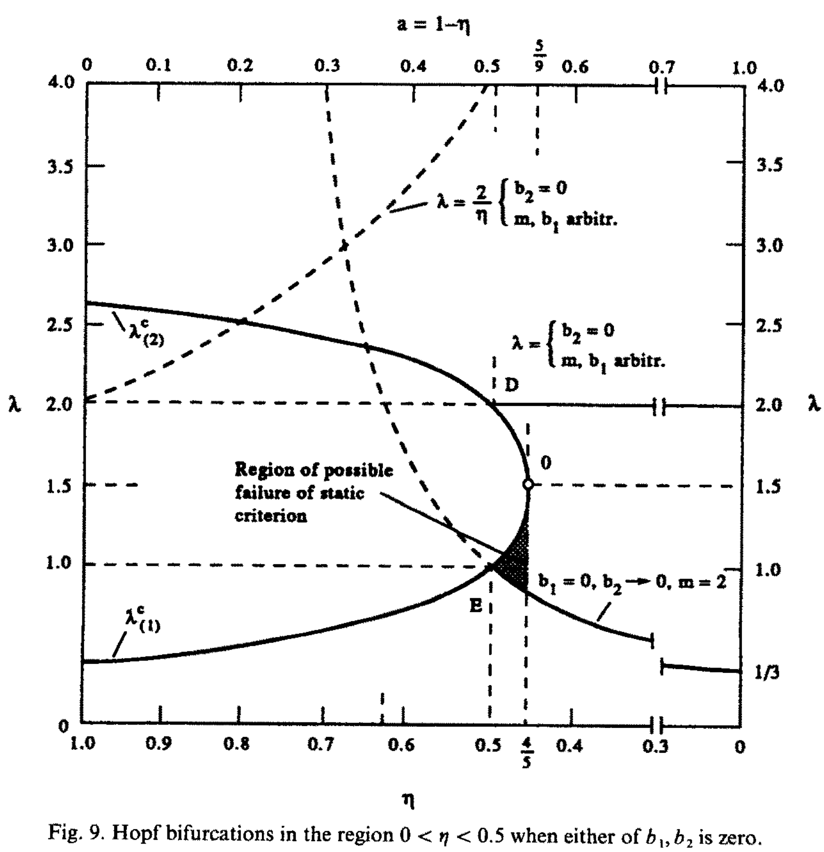

Fig. 9. Hopf bifurcations in the region $0 < \eta < 0.5$ when either of b_1, b_2 is zero.

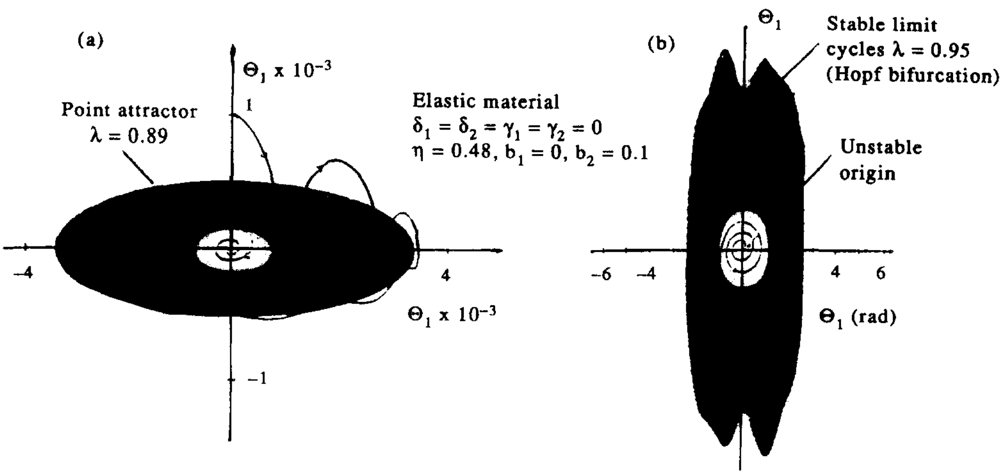

Fig. 10. Point attractor for $\lambda = 0.89$ (a) and stable limit cycles (b) for $\lambda = 0.95$ for $\eta = 0.48$, $b_1 = 0$, $b_2 = 0.1$.

the undamped system. The classical analysis for $\eta = 0$ leads to a divergent motion, while the present nonlinear analysis shows that the motion is bounded (Kounadis, 1990).

Figure 10 shows two phase–plane portraits corresponding to $\lambda = 0.89$ (point attractor) and $\lambda = 0.95$ (Hopf bifurcation) for an elastic damped model in which dynamic instability occurs prior to divergence, since $\lambda = 0.95 < \lambda^c_{(1)} = 1.09175$. Figure 11 shows the stable limit cycles of the compound branching (pseudo-equilibrium) point (η_0, λ^c_0).

Besides local (Hopf) bifurcations one can discover global dynamic bifurcations using a nonlinear dynamic analysis. The change of their phase portrait is not noticeable in the neighborhood of any equilibrium point, but can only be discerned on a global scale. Global bifurcations are not detected in the region of nonexistence of adjacent equilibria ($\eta < 4/9$), but only in the region of existence of such equilibria occurring always for loads $\lambda > \lambda_{(1)}$ (i.e. after divergence instability). They start to appear for $\eta > 4/9$ and their locus extends a little beyond $\eta = 0.50$ (Fig. 12). A typical phase–plane portrait is shown in Fig. 13.

6. CONCLUSIONS

Using a qualitative and quantitative analysis, the neighborhood of a compound branching point, being the boundary between static and dynamic instability, is thoroughly discussed. The following findings are worthy of report:

(1) In the vicinity of this point there exists a small region of adjacent equilibria, where the loss of stability in cases of vanishing damping may occur via a Hopf (dynamic)

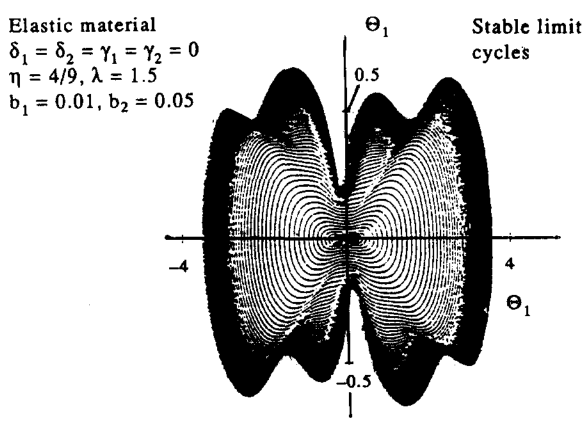

Fig. 11. The phase-portrait of the point ($\eta_0 = 4/9$, $\lambda^c_0 = 1.5$) associated with an unstable origin and stable limit cycles for $b_1 = 0.01$, $b_2 = 0.05$.

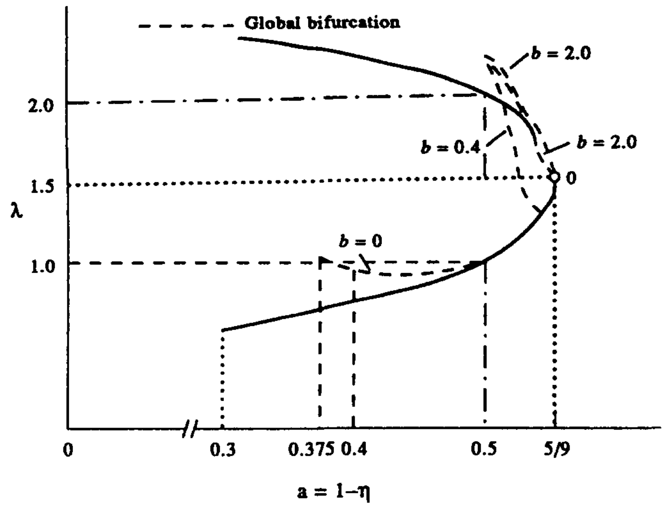

Fig. 12. Loci of global bifurcations corresponding to various b ($=0, 0.4, 1, 2$).

bifurcation prior to static (divergence) buckling. Hence, the static criterion fails to predict the actual critical load. If the vanishing damping is replaced by zero damping, the critical load takes its maximum value coinciding with that of divergence instability.

(2) In this region, which is explicitly determined, in addition to the above findings the following phenomena are also found:

- An interaction of two consecutive postbuckling modes.
- The critical states of static (divergence) instability may be associated with a double zero eigenvalue also satisfying the conditions of Hopf (local) bifurcations.
- While the system is perfect with trivial precritical deformation, its critical load is strongly affected by both the mass distribution and damping ratio.
- The compound branching point is a hybrid or pseudo-equilibrium point since its response is associated with limit cycles.
- As we pass from the region of adjacent equilibria to the region of nonexistence of such equilibria there may be a loading discontinuity with a flutter load lower than, equal to or higher than the static buckling load (contrary to the corresponding finding of the classical analysis) depending on the damping ratio.
- In the region of adjacent equilibria, in addition to local (Hopf) bifurcations, there exist global bifurcations established only by nonlinear dynamic analysis. However, these bifurcations occur for loads higher than the critical ones.

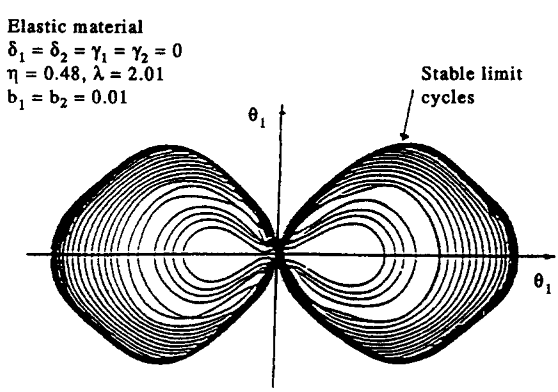

Fig. 13. Global stable bifurcation with trajectories passing through the saddle of the origin for $\eta = 0.48$, $\lambda = 2.01 > \lambda_{cr} = 2.007$, $b_1 = b_2 = 0.01$.

- Regardless of whether or not the system loses its stability via divergence or flutter, the postcritical response is associated either with a point or a limit cycle attractor. This contradicts the classical analysis which leads to a divergent motion in both cases.

(3) The postbuckling paths for $\eta > 0.5$ are independent of each other and the system exhibits a point attractor.

(4) The loss of stability in the region of nonexistence of adjacent equilibria always occurs via a Hopf bifurcation.

- The critical flutter load has as upper bound the critical load of the corresponding undamped system.

(5) There is a considerable variation of the flutter load depending on the mass and damping ratio in both regions.

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